Exam

Statistical Physics

Thursday January 26, 2017
9:00-12:00

Read these instructions carefully before making the exam!

- **Write your name and student number on *every* sheet.**
- ***Make sure to write readable for other people than yourself. Points will NOT be given for answers in illegible writing.***
- ***Language*; your answers have to be in English.**
- **Use a *separate* sheet of paper for each problem.**
- **Use of a (graphing) calculator is allowed.**
- **This exam consists of 4 problems.**
- **The weight of the problems is: Problem 1 (P1=25 pts); Problem 2 (P2=20 pts); Problem 3 (P3=20 pts); Problem 4 (P4=25 pts). Weights of the various subproblems are indicated at the beginning of each problem.**
- **The grade of the exam is calculated as (P1+P2+P3+P4+10)/10.**
- **For all problems you have to write down your arguments and the intermediate steps in your calculation, *else the answer will be considered as incomplete and points will be deducted.***

PROBLEM 1

Score: a+b+c+d+e+f=4+4+4+4+5+4=25

Consider a solid that consists of a large number (N) of atoms with spin $\frac{1}{2}$, each of which has a fixed position in space. Each atom has a magnetic moment μ that can be aligned either parallel or anti-parallel with an external magnetic field B. We assume that the magnetic moment of one atom interacts only very weakly with those around it. The solid is in equilibrium at temperature T.

The energy levels of a single atom are:

$\varepsilon_1 = -\mu B$: if the spin is parallel to the magnetic field B

$\varepsilon_2 = \mu B$: if the spin is antiparallel to the magnetic field B

We define the variable, $x = \frac{\mu B}{kT}$.

a) Give the single-atom partition function z (express your answer in terms of x).

b) Give expressions for the probabilities P_1 and P_2 that the energy levels ε_1 and ε_2 are occupied (express your answers in terms of x). Determine the values of P_1 and P_2 in the limit $T \to 0$ and in the limit $T \to \infty$.

c) Explain why the partition function Z for N spins is given by $Z = z^N$ and does not need a factor $1/N!$.

d) Use the partition function Z to show that the total energy E due to the spins of the N atoms is given by,

$$E = -NkTx\frac{e^x - e^{-x}}{e^x + e^{-x}}$$

e) Prove that the entropy S of the spins of the N atoms is given by,

$$S = Nk\left(\ln[e^x + e^{-x}] - x\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$$

f) Suppose the solid is in equilibrium at a temperature $T_1 = 10$ K. The magnetic field is then reduced from its initial value B_1 to a lower value $B_2 = 0.01B_1$ in a *reversible adiabatic* process. Calculate the temperature T_2 of the system after this process.

PROBLEM 2

Score: a+b+c+d+e=4+4+4+4+4=20

A system consists of two identical non-interacting particles. The system has three single-particle states: φ_1, φ_2 and φ_3 with energies $\varepsilon_1 = 0 < \varepsilon_2 < \varepsilon_3$, respectively.

Use the notation (n_1, n_2, n_3) for two-particle states with n_i $(i = 1,2,3)$ the occupation number of single-particle state φ_i $(i = 1,2,3)$, respectively. For example: (1,1,0) means that one particle is in state φ_1 and one particle is in state φ_2.

List all two-particle states with their energy and degeneracy in case the two particles are:

a) Distinguishable classical particles
b) Indistinguishable fermions
c) Indistinguishable bosons
d) Give the partition function for situation a), b) and c).
e) Using only the two leading terms for low temperatures in the partition function find the energy as function of the temperature for situation b) and c). Compare the behaviour of the two situations in the limit $T \to 0$.

PROBLEM 3

Score: a+b+c+d+e=5+3+3+4+5=20

A classical perfect gas of N atoms with mass m is confined to two dimensions (surface $A = L_x L_y$). The number of atoms, N, is a very large number.

a) Show that the single-particle partition function Z_1 of this gas is given by,

$$Z_1 = A\left(\frac{2\pi mkT}{h^2}\right)$$

HINT: The density of states for a *spinless* particle confined to a 2-dimensional enclosure with surface area A is (expressed as a function of the particle's momentum p):

$$f(p)dp = \frac{A}{h^2}2\pi p dp$$

b) Show that the Maxwell speed distribution of this 2D gas is,

$$P(v)dv = \frac{mv}{kT}e^{-\frac{mv^2}{2kT}}dv$$

c) Calculate the average speed of the atoms.

d) Show that the Helmholtz free energy F for this gas is given by:

$$F = -NkT\left(\ln\left(\frac{A2\pi mkT}{Nh^2}\right) + 1\right)$$

e) Use F to derive the equation of state of this gas.

PROBLEM 4

Score: a+b+c+d+e =6+5+4+5+5=25

A gas of photons is confined to a cavity with volume V. The cavity is kept at a constant temperature T.

HINT 1: The density of states for a *spinless* particle confined to an enclosure with volume V is (expressed as a function of the particle's momentum p):

$$f(p)dp = \frac{V}{h^3} 4\pi p^2 dp$$

HINT 2: The mean number of photons in a state with energy $\varepsilon = \hbar\omega$ is equal to: $\frac{1}{e^{\beta\varepsilon}-1}$

a) Show that density of states of a photon in the cavity can be written as,

$$f(\omega)d\omega = \frac{V\omega^2 d\omega}{\pi^2 c^3}$$

b) Show that the total number of photons in the cavity is given by,

$$N = b\,\frac{Vk^3T^3}{\pi^2\hbar^3c^3}$$

where b is dimensionless constant.

c) Find the numerical value of b.

d) Show that the total energy density $u = \frac{E}{V}$ (J m^{-3}) in the cavity is related to the temperature T by,

$$u = aT^4 \text{ with } a = \frac{\pi^2 k^4}{15\hbar^3 c^3}$$

The Helmholtz free energy F of the photon gas is $F = -\frac{1}{3}aVT^4$.

e) Prove that for the photon gas the energy E is given by,

$$E = F + TS$$

with S the entropy of the photon gas.

Solutions

PROBLEM 1

a)
Use the definition of the partition function: Eq. 2.23 (Mandl)

$$z = e^{-\beta\varepsilon_1} + e^{-\beta\varepsilon_2} = e^{\beta\mu B} + e^{-\beta\mu B} = e^x + e^{-x}$$

b)
Probability that an atom is in state ε_1 is :

$$P_1 = \frac{e^{-\beta\varepsilon_1}}{z} = \frac{e^x}{e^x + e^{-x}}$$

Probability that an atom is in state ε_2 is :

$$P_2 = \frac{e^{-\beta\varepsilon_2}}{z} = \frac{e^{-x}}{e^x + e^{-x}}$$

Limit $T \to 0$ then $\beta \to \infty$ and $x \to \infty$ and we have $P_1 \to 1$ and $P_2 \to 0$, all spins are in the low-energy state and are aligned with the magnetic field.

Limit $T \to \infty$ then $\beta \to 0$ and $x \to 0$ and we have $P_1 \to \frac{1}{2}$ and $P_2 \to \frac{1}{2}$, the spins are evenly distributed over both energy states.

c)
The spins are at a fixed position in the solid and are thus distinguishable by their position which implies that we do not need to correct for permutations for indistinguishable particles.

d)

The energy E can be calculated from the partition function by,

$$E = -\frac{\partial \ln Z}{\partial \beta} = -\frac{\partial \ln Z}{\partial x}\frac{\partial x}{\partial \beta} = -\mu B\frac{\partial \ln Z}{\partial x} = -\mu B\frac{\partial \ln z^N}{\partial x} = -N\mu B\frac{\partial \ln z}{\partial x} = \frac{-N\mu B}{z}\frac{\partial z}{\partial x}$$

and with $\frac{\partial z}{\partial x} = e^x - e^{-x}$ we find

$$E = -N\mu B \frac{e^x - e^{-x}}{e^x + e^{-x}} = -NkTx \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Alternatively, the mean energy of one atom is,

$$\bar{\varepsilon} = p_1\varepsilon_1 + p_2\varepsilon_2 = \frac{\varepsilon_1 e^x}{e^x + e^{-x}} + \frac{\varepsilon_2 e^{-x}}{e^x + e^{-x}} = \frac{-\mu B e^x + \mu B e^{-x}}{e^x + e^{-x}} = -kTx \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Thus for the N atoms,

$$E = N\bar{\varepsilon} = -NkTx \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

d)

The Helmholtz free energy is given by,

$$F = -kT \ln Z = -kT \ln z^N = -NkT \ln z = -NkT \ln[e^x + e^{-x}]$$

$$F = E - TS \Rightarrow S = \frac{E - F}{T} = -Nkx \frac{e^x - e^{-x}}{e^x + e^{-x}} + Nk \ln[e^x + e^{-x}] \Rightarrow$$

$$S = Nk \left(\ln[e^x + e^{-x}] - x \frac{e^x - e^{-x}}{e^x + e^{-x}} \right)$$

e)

In the reversible adiabatic process the entropy does not change, thus

$$S(T_1, B_1) = S(T_2, B_2)$$

This implies,

$$x_1 = x_2 \Rightarrow \frac{\mu B_1}{kT_1} = \frac{\mu B_2}{kT_2} \Rightarrow T_2 = \frac{B_2}{B_1} T_1 = 0.01 \times 10 = 0.1 \text{ K}$$

PROBLEM 2

a) (1,1,0) with $\varepsilon = \varepsilon_2$, and degeneracy: 2
(0,1,1) with $\varepsilon = \varepsilon_2 + \varepsilon_3$, and degeneracy: 2
(1,0,1) with $\varepsilon = \varepsilon_3$, and degeneracy: 2
(2,0,0) with $\varepsilon = 0$, and degeneracy: 1
(0,2,0) with $\varepsilon = 2\varepsilon_2$ and degeneracy:1
(0,0,2) with $\varepsilon = 2\varepsilon_3$ and degeneracy:1

Remark, because the particles are distinguishable interchanging the two particles gives a different state if the particles are in different single-energy states.

b) (1,1,0) with $\varepsilon = \varepsilon_2$, and degeneracy: 1
(0,1,1) with $\varepsilon = \varepsilon_2 + \varepsilon_3$, and degeneracy: 1
(1,0,1) with $\varepsilon = \varepsilon_3$, and degeneracy: 1

Remark, indistinguishable implies that also the spin variable is the same! So we cannot have two fermions in the same state with opposite spins.

c) (1,1,0) with $\varepsilon = \varepsilon_2$, and degeneracy: 1
(0,1,1) with $\varepsilon = \varepsilon_2 + \varepsilon_3$, and degeneracy: 1
(1,0,1) with $\varepsilon = \varepsilon_3$, and degeneracy: 1
(2,0,0) with $\varepsilon = 0$, and degeneracy: 1
(0,2,0) with $\varepsilon = 2\varepsilon_2$ and degeneracy:1
(0,0,2) with $\varepsilon = 2\varepsilon_3$ and degeneracy:1

d)
Situation a) $Z = 2e^{-\beta\varepsilon_2} + 2e^{-\beta(\varepsilon_2+\varepsilon_3)} + 2e^{-\beta\varepsilon_3} + 1 + e^{-\beta 2\varepsilon_2} + e^{-\beta 2\varepsilon_3}$
Situation b) $Z = e^{-\beta\varepsilon_2} + e^{-\beta(\varepsilon_2+\varepsilon_3)} + e^{-\beta\varepsilon_3}$
Situation c) $Z = e^{-\beta\varepsilon_2} + e^{-\beta(\varepsilon_2+\varepsilon_3)} + e^{-\beta\varepsilon_3} + 1 + e^{-\beta 2\varepsilon_2} + e^{-\beta 2\varepsilon_3}$

e)
Situation b) $Z \approx e^{-\beta\varepsilon_2} + e^{-\beta\varepsilon_3}$
And

$$\bar{\varepsilon} = -\frac{\partial \ln Z}{\partial \beta} = \frac{\varepsilon_2 e^{-\beta\varepsilon_2} + \varepsilon_3 e^{-\beta\varepsilon_3}}{e^{-\beta\varepsilon_2} + e^{-\beta\varepsilon_3}}$$

In the limit $T \to 0$

$$\bar{\varepsilon} = \frac{\varepsilon_2 e^{-\beta\varepsilon_2} + \varepsilon_3 e^{-\beta\varepsilon_3}}{e^{-\beta\varepsilon_2} + e^{-\beta\varepsilon_3}} = \frac{\varepsilon_2 + \varepsilon_3 e^{-\beta(\varepsilon_3-\varepsilon_2)}}{1 + e^{-\beta(\varepsilon_3-\varepsilon_2)}} = \frac{\varepsilon_2 + \varepsilon_3 e^{-\beta(\varepsilon_3-\varepsilon_2)}}{1 + e^{-\beta(\varepsilon_3-\varepsilon_2)}} \frac{\left(1 - e^{-\beta(\varepsilon_3-\varepsilon_2)}\right)}{\left(1 - e^{-\beta(\varepsilon_3-\varepsilon_2)}\right)} \Rightarrow$$

$$\bar{\varepsilon} = \frac{\varepsilon_2 + (\varepsilon_3 - \varepsilon_2) e^{-\beta(\varepsilon_3-\varepsilon_2)} - \varepsilon_3 \left(e^{-\beta(\varepsilon_3-\varepsilon_2)}\right)^2}{1 - (e^{-\beta(\varepsilon_3-\varepsilon_2)})^2} \approx \varepsilon_2 + (\varepsilon_3 - \varepsilon_2) e^{-\beta(\varepsilon_3-\varepsilon_2)}$$

Situation c) $Z = 1 + e^{-\beta\varepsilon_2}$

$$\bar{\varepsilon} = -\frac{\partial \ln Z}{\partial \beta} = \frac{\varepsilon_2 e^{-\beta\varepsilon_2}}{1 + e^{-\beta\varepsilon_2}}$$

In the limit $T \rightarrow 0$

$$\bar{\varepsilon} \approx \varepsilon_2 e^{-\beta\varepsilon_2}$$

In case $T = 0$, the two-fermion system has an energy larger than zero (ε_2); the two-boson system has zero energy.

PROBLEM 3

a)
The density of states for one atom in 2D is:

$f(p)dp = \frac{A}{h^2} 2\pi p dp$

The partition function of this atom is then:

$$Z_1 = \int_0^\infty f(p) e^{-\frac{\beta p^2}{2m}} dp = \int_0^\infty \frac{A}{h^2} 2\pi p e^{-\frac{\beta p^2}{2m}} dp$$

Introduce $x^2 = \frac{\beta p^2}{2m} \Rightarrow x = \sqrt{\frac{\beta}{2m}} p \Rightarrow p = \sqrt{\frac{2m}{\beta}} x$ to convert the integral to,

$$Z_1 = \frac{2\pi A}{h^2} \left(\sqrt{\frac{2m}{\beta}} \right)^2 \int_0^\infty x e^{-x^2} dx = \frac{2\pi A}{h^2} \left(\sqrt{\frac{2m}{\beta}} \right)^2 \left(\frac{1}{2} \right) = A \left(\frac{2\pi m k T}{h^2} \right)$$

The integral is in the list of Physical constants and Integrals

b)
The probability $P(p)$ to find an atom with momentum between p and $p + dp$ is proportional to $f(p) e^{-\frac{\beta p^2}{2m}}$, thus

$$P(p)dp \propto \frac{A}{h^2} 2\pi p e^{-\frac{\beta p^2}{2m}} dp \Rightarrow P(p)dp = C \frac{A}{h^2} 2\pi p e^{-\frac{\beta p^2}{2m}} dp$$

with C the proportionality constant, and because we should have,

$$\int_0^\infty P(p) dp = 1$$

we find,

$$\int_0^\infty C \frac{A}{h^2} 2\pi p e^{-\frac{\beta p^2}{2m}} dp = 1 \Rightarrow C = \frac{1}{Z_1}$$

Thus,

$$P(p)dp = \frac{1}{Z_1}\frac{A}{h^2}2\pi p e^{-\frac{\beta p^2}{2m}}dp = \left(\frac{h^2}{2\pi mkT}\right)\frac{1}{h^2}2\pi p e^{-\frac{\beta p^2}{2m}}dp$$

And converting to speed v by using $p = mv$ we find,

$$P(v)dv = \left(\frac{mv}{kT}\right)e^{-\frac{mv^2}{2kT}}dv$$

c) Average speed (mean speed) is found by,

$$\overline{v} = \int_0^\infty vP(v)dv = \left(\frac{m}{kT}\right)\int_0^\infty v^2 e^{-\frac{mv^2}{2kT}}dv$$

Define $u^2 = \frac{mv^2}{2kT} \Rightarrow u = \frac{v}{\sqrt{\frac{2kT}{m}}} \Rightarrow v = \sqrt{\frac{2kT}{m}}u$ and substitute,

$$\overline{v} = \left(\frac{m}{kT}\right)\left(\frac{2kT}{m}\right)^{3/2}\int_0^\infty u^2 e^{-u^2}du = \left(\frac{m}{kT}\right)\left(\frac{2kT}{m}\right)^{3/2}\left(\frac{1}{4}\sqrt{\pi}\right) = \sqrt{\left(\frac{\pi kT}{2m}\right)}$$

The integral is in the list of Physical constants and Integrals.

d)

The partition function of the 2D classical gas is:

$$Z_N = \frac{1}{N!}(Z_1)^N = \frac{1}{N!}A^N\left(\frac{2\pi mkT}{h^2}\right)^N$$

The Helmholtz free energy F is:

$$F = -kT\ln Z_N = -kT(N\ln Z_1 - N\ln N + N) = -kT\left(N\ln\frac{A2\pi mkT}{h^2} - N\ln N + N\right) \Rightarrow$$

$$F = -NkT\left(\ln\left(\frac{A2\pi mkT}{Nh^2}\right) + 1\right)$$

In this derivation Stirling's approximation was used ($\ln N! = N\ln N - N$).

e)

Using $F = E - TS$ we have (Use A instead of V for a 2D gas)

$$dF = dE - TdS - SdT = TdS - PdA - TdS - SdT = -PdA - SdT$$

And consequently

$$P = -\left(\frac{\partial F}{\partial A}\right)_T$$

With the expression from d) this leads to

$$P = \frac{NkT}{A} \Rightarrow PA = NkT$$

PROBLEM 4

a)
For photons the momentum p is related to energy $\varepsilon = \hbar\omega = pc$. Using this in HINT 1 in combination with the fact that the photon has two polarization states leads to,

$$f(\omega)d\omega = 2\frac{V}{h^3}4\pi\left(\frac{\hbar\omega}{c}\right)^2 d\left(\frac{\hbar\omega}{c}\right) = \frac{V}{\pi^2\hbar^3}\left(\frac{\hbar}{c}\right)^3\omega^2 d\omega = \frac{V\omega^2 d\omega}{\pi^2 c^3}$$

b)
Using the density of states in a) and the mean number of photons in each state $n(\omega)$ (from HINT 2) we find,

$$N = \int_0^\infty n(\omega)f(\omega)d\omega = \int_0^\infty \frac{1}{e^{\beta\hbar\omega}-1}\frac{V\omega^2 d\omega}{\pi^2 c^3}$$

With the substitution $x = \beta\hbar\omega$ this leads to,

$$N = \frac{V}{\pi^2 c^3}\frac{1}{(\beta\hbar)^3}\int_0^\infty \frac{x^2 dx}{e^x - 1} = b\frac{Vk^3T^3}{\pi^2\hbar^3 c^3}$$

c)
Use the table with the integrals and constants,

$$b = \int_0^\infty \frac{x^2 dx}{e^x - 1} = 2.404$$

d)
The total energy E in the cavity is,

$$E = \int_0^\infty \hbar\omega\, n(\omega)f(\omega)d\omega = \int_0^\infty \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\frac{V\omega^2 d\omega}{\pi^2 c^3}$$

Again using the substitution $x = \beta\hbar\omega$ this leads to,

$$E = \frac{\hbar V}{\pi^2 c^3}\frac{1}{(\beta\hbar)^3}\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{Vk^4T^4}{\pi^2\hbar^2 c^3}\int_0^\infty \frac{x^3 dx}{e^x - 1}$$

The integral is equal to $\frac{\pi^4}{15}$, this follows from the table with the integrals and constants, thus,

$$E = \frac{Vk^4T^4}{\pi^2\hbar^2c^3}\frac{\pi^4}{15} = \frac{V\pi^2k^4T^4}{15\hbar^2c^3} \Rightarrow$$

$$u = \frac{E}{V} = \frac{\pi^2k^4T^4}{15\hbar^2c^3} = aT^4$$

e)
Helmholtz Free energy is given by $F = -\frac{1}{3}aVT^4$. The entropy is,

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = \frac{4}{3}aVT^3$$

Thus,

$$E = F + TS = -\frac{1}{3}aVT^4 + \frac{4}{3}aVT^3T = aVT^4$$